3-MANIFOLDS FROM PLATONIC SOLIDS

B. EVERITT

ABSTRACT. The problem of classifying, upto isometry (or similarity), the orientable spherical, Euclidean and hyperbolic 3-manifolds that arise by identifying the faces of a Platonic solid is formulated in the language of Coxeter groups. In the spherical and hyperbolic cases, this allows us to complete the classification begun by Lorimer [11], Richardson and Rubinstein [17] and Best [2].

1. INTRODUCTION

The first example of an orientable hyperbolic 3-manifold arose by identifying the faces of a solid hyperbolic dodecahedron [20]. Of course in the intervening years, much much more has been said about such manifolds. Yet, the classical question of which spherical, Euclidean or hyperbolic manifolds arise by identifying the faces of a Platonic solid has a surprisingly incomplete solution.

In this paper we formulate the problem in terms of classifying certain subgroups of rank four Coxeter groups. This approach is implicit in [11, 17], and this paper should be viewed as completing their work in the spherical and hyperbolic cases. It follows an earlier, oft quoted but flawed attempt in [2]. As the results of [17] are not readily available in the literature, we summarise them in Table 2.

There are other, non-algebraic, approaches to the problem, particularly that of Molna'r and his school (see for instance [12, 14, 15] and the references there). In fact, our list of manifolds in the Euclidean case cannot be given precisely without recourse to Prok's paper [15]. The author is very grateful to Colin Maclachlan, Emil Molna'r, Istvan Prok, Peter Lorimer and Marston Conder for many useful discussions and suggestions. He also thanks Hyam Rubinstein for a copy of the preprint [17].

2. PLATONIC SOLIDS AND COXETER GROUPS

Let $X = S^3, \mathbb{E}^3$ or $\mathbb{H}^3$, and suppose $\Delta \subset X$ is a finite volume Coxeter simplex (see [10]) with symbol,

$$\circ\overset{p}{\text{---}}\circ\overset{q}{\text{---}}\circ\overset{r}{\text{---}}\circ \tag{1}$$

Each node of the symbol corresponds to a face of Δ, which in turn has a vertex of Δ opposite it. Call this the vertex corresponding to the node. Let $\Gamma = \{p, q, r\}$ be the Coxeter group generated by reflections in the faces of Δ, and for any vertex, edge or face of Δ, say $*$, let Γ_* be its stabiliser in Γ. In particular, if v is a vertex of Δ, then Γ_v is also Coxeter group, its symbol obtained from (1) by deleting the node corresponding to v and its incident edges.

Let v be the vertex of Δ corresponding to the left-most node of (1). Then,

$$\Sigma = \bigcup_{\gamma \in \Gamma_v} \gamma(\Delta), \tag{2}$$

is a solid with r-gonal faces, q meeting at each vertex, and dihedral angle (that is, angle subtended by adjacent faces) $2\pi/p$. Similarly for the last node with corresponding vertex v', from which we obtain a solid Σ' with p-gonal faces, q meeting at each vertex and dihedral angle $2\pi/r$. The tessellations of X by

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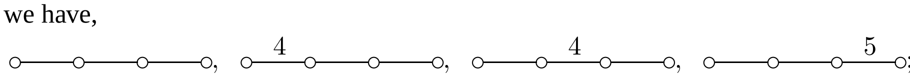
N	FI	EI	H_1
1	abcdefefbcda	a(-+)b(-+)c(-+)d(-+)e(-+)f(-+)g(-+)	00000
		h(-+)i(-+)j(-+)idjefagbhcghijfeabcd	
2	abcdefbdcfea	a(-+)b(-+)c(-+)d(-+)e(-+)f(++)g(++)	(15)0000
		h(++)i(++)j(++)ajcgbfeidhfhgjieabcd	

TABLE 1. The spherical manifolds arising from a dodecahedron with dihedral angle $2\pi/3$, [11].

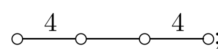
congruent copies of Σ and Σ' that result from successive reflections in the faces of these solids are duals to one another, and both have automorphism group Γ.

On the otherhand, suppose we have a Platonic solid in X. By this we mean a polytope P with the combinatorial type of a Platonic solid (convex regular solid), embedded in X so that all side lengths are equal, as are the interior face angles and dihedral angles. For face identifications of P to yield an X-manifold, the dihedral angle must be a submultiple of 2π, say $2\pi/p$. Barycentric subdividion of P then gives a Coxeter simplex of the type (1), and P is recoverable in the form (2) using the vertex v of the simplex lying at the center of P. Thus, the problem of obtaining manifolds from a general Platonic solid P reduces to consideration of the Σ obtained at (2).

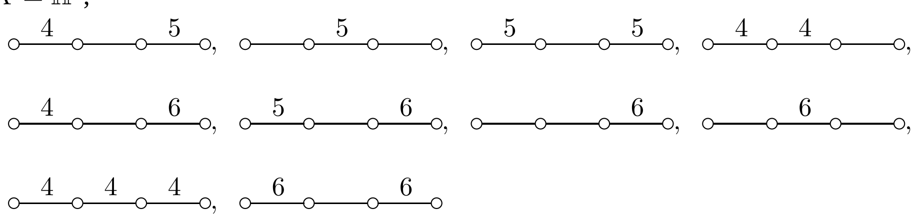
All Coxeter simplices of the form (1) are known and listed in Sections 2.4, 2.5 and 6.9 of [10]. For $X = S^3$ we have,



for $X = \mathbb{E}^3$ we get



and for $X = \mathbb{H}^3$,



In the spherical case, the tessellations of S^3 by copies of Σ or Σ' give the six 4-dimensional regular solids [8]. In another incarnation, the first three give Γ that are the Weyl groups of the Lie algebras of type $A_4 = \mathfrak{sl}_5(\mathbb{C})$, $B_4 = \mathfrak{so}_9(\mathbb{C})$ and F_4. The hyperbolic Γ give Σ and Σ' of finite volume: the first three compact, the others non-compact.

We get a total of six spherical, one Euclidean and eight hyperbolic Platonic solids from these groups: spherical tetrahedra with dihedral angles $2\pi/3, 2\pi/4$ and $2\pi/5$, a cube with angle $2\pi/3$, an octahedron with angle $2\pi/3$ and a dodecahedron with angle $2\pi/3$; in the Euclidean case we get the familiar cube; and in the hyperbolic, a compact octahedron, icosahedron and two dodecahedrons with angles $2\pi/5, 2\pi/3, 2\pi/4$ and $2\pi/5$; finally, a non-compact but finite volume cube, octahedron, dodecahedron and tetrahedron with dihedral angles $2\pi/4, 2\pi/6, 2\pi/6$ and $2\pi/6$ respectively.

3. CONSTRUCTING THE MANIFOLDS

Any X-manifold (see [18, §3.3]) arises as the quotient X/K of X by a group K acting properly discontinuously and without fixed points. When $X = \mathbb{E}^3$ or $\mathbb{H}^3$, the isometries of X with fixed points are precisely those of finite order. This allows a simple algebraic formulation of the problem in these two geometries (Theorem 1 below). Alternatively, recourse to a more geometric view yields Theorem 2,

which holds for all three geometries. The statements in the remainder of the paper will be formulated in terms of the solid Σ, those for Σ' being entirely analogous.

Establishing first some notation, let $\mathfrak{S}_m$ be the symmetric group of degree m. If Λ is a subgroup of $\mathfrak{S}_m$, let Λ_i be the stabiliser in Λ of $i \in \{1, \dots, m\}$. For any group G, let $\mathscr{T}(G)$ be a subset that contains *at least one* representative from each conjugacy class of elements of finite prime order.

Theorem 1. Let $X = \mathbb{E}^n$ or $\mathbb{H}^n$ for $n \geq 2$; Γ a group acting properly discontinuously by isometries on X with (convex, locally finite) fundamental region P; F a finite subgroup of Γ and

$$\Sigma = \bigcup_{\gamma \in F} \gamma(P).$$

An X-manifold M arises by the identification of points on the boundary of Σ if and only if there is a homomorphism $\varepsilon : \Gamma \to \mathfrak{S}_m$, where m is the order of F, such that,

1. if $\Lambda = \varepsilon(\Gamma)$, then Λ acts transitively on $\{1, \dots, m\}$, and
2. for all $\gamma \in \mathscr{T}(\Gamma)$, the permutation $\varepsilon(\gamma)$ fixes no point of $\{1, \dots, m\}$.

Moreover, if $i \in \{1, \dots, m\}$, then $\pi_1(M) \cong \varepsilon^{-1}(\Lambda_i)$.

Proof. An X-manifold M arises by identifying points on $\partial\Sigma$ if and only if there is a torsion free subgroup K of Γ with fundamental region Σ and M isometric to the quotient X/K. Such a K (which is isomorphic to $\pi_1(M)$) may be replaced by any of its conjugates in Γ, as these will yield quotients isometric to M. Conjugacy classes of subgroups of Γ of index m correspond to transitive actions of Γ on $\{1, \dots, m\}$, the subgroups arising as the stabilisers of points. These actions in turn correspond to homomorphisms $\Gamma \to \mathfrak{S}_m$ with transitive image.

A subgroup K is torsion free if and only if it intersects trivially the conjugacy class in Γ of each $\gamma \in \mathscr{T}(\Gamma)$. This happens precisely when $\varepsilon(\gamma)$ has no fixed points among $\{1, \dots, m\}$. Finally, Σ forms a fundamental region for the action of K on X exactly when F forms a transversal (a non-redundant list of coset representatives) for K in Γ. Equivalently, $K \cap F = \{1\}$ and $KF = \Gamma$. The first follows immediatly as K is torsion free, and the second, since F is a subgroup, when the index of K in Γ is equal to the order of F. □

We will be applying Theorem 1 with F the stabiliser Γ_v. In an arbitrary Coxeter group Γ, a $\mathscr{T}(\Gamma)$ can be found using [5, 9]–list for example the conjugacy class representatives in the maximal finite parabolic subgroups of Γ. For the group with symbol (1), or in fact for any 3-dimensional Euclidean or hyperbolic Coxeter group, it is particularly easy to find a $\mathscr{T}(\Gamma)$: take the generating reflections and the powers of their pairwise products that have prime order.

More geometrically, suppose we have a subgroup K of Γ for which Γ_v is a transversal, and let S be a face of Σ. In the tessellation of X by copies of Σ there is a unique copy Σ_S of Σ with $\Sigma \cap \Sigma_S = S$. Since Σ forms a fundamental region for K, there is a unique element $\gamma_S \in K$ sending Σ to Σ_S, and hence a face S' of Σ with $\gamma_S(S') = S$. The collection of isometries $\{\gamma_S\}_{S \in \Sigma}$ yield a side-pairing of Σ as in [16, Section 10.1]. It follows immediately from Theorems 10.1.2 and 10.1.3 of [16] that,

Theorem 2. Let $X = S^3, \mathbb{E}^3$ or $\mathbb{H}^3$. An X-manifold M arises by the identification of faces of (2) if and only if Γ has a subgroup K of orientation preserving isometries, such that

1. Γ_v forms a transversal in Γ for K;
2. if $\{\gamma_S\}$ are the resulting side pairings of Σ, then γ_S fixes no point of S'; and
3. for $x \in \Sigma$, let $[x]$ denote the points of Σ identified with it under the side pairing. If x lies in the interior of an edge of Σ, then $[x]$ has cardinality p.

So we merely require that the faces of Σ are identified in pairs and the edges in groups of p. The identifications can be described algebraically as follows: since Γ acts transitively on the k-cells ($k = 0, 1, 2, 3$) of the tessellation of X by Σ, the faces of Σ are in one to one correspondence with the cosets

N	FI	EI	H_1
1	abcdefefbcda	a(-+-+)b(-+-+)c(-+-+)d(-+-+)e(-+-+) cdeabf(++++)afbfcfdfecdeabdeabc	55500
2	abcdefdefbca	a(++++)b(++++)c(++++)d(++++)e(++++) abcdebf(++++)cfdfefafcdeabbcdea	55500
3	abcdefdefbca	a(+-++)b(-+++)c(---+)d(++-+)e(+-++) debaf(+-++)bcfafefcdcfedabeabcd	33000
4	abccadeefbfd	a(++--)ab(-+++)ac(-+-+)d(-+++)bab e(+++-)ef(--+-)bfdcaecdfffddcbece	57000
5	abcdefebfdca	a(-+-+)b(-+-+)c(-+-+)d(-+-+)e(-+-+) edacbf(++++)cfefbfafdbdaeceabcd	35500
6	abcdeffbdeca	a(++++)b(++++)c(++++)d(++++)e(++++) cf(++++)efdfbfafeacdbdeabc	33550
7	abcdebedffca	a(+-++)b(+-++)c(---+)d(-+-+)e(-+++) cedaef(--+-)afdfbfcfebdcbacdeab	3(16)000
8	abbcadefecfd	a(+++-)b(++-+)c(--++)ad(-++-)a e(+-+-)dbbeadcf(+--+)acfceffdedbdbfc	(29)0000
9	abcbdaefghihdefjgcji	a(-+)b(+-)c(--)d(-+)e(-+)deabf(++) g(+-)h(-+)i(+-)iaccj(++)jhdebfgfghij	(11)(11)000
10	abcdebfceghhiijjfgda	a(-+)b(-+)c(-+)d(--)e(++)cf(--)ea g(--)ebh(+-)gi(++)dj(+-)fghhdiifjjabc	90000
11	abcdefbdgehiijjhfgca	a(++)b(++)c(++)d(++)e(+-)cdf(+-)ad g(+-)bfh(-+)gi(+-)ej(-+)ijgjhehifabc	22900
12	abcdaefdgfhihcjjbige	a(++)b(+-)bc(+-)d(--)e(+-)baf(--) g(+-)efgh(++)ghci(+-)dj(-+)jjdeiicahf	57000
13	abcdabefghcijidfjghe	a(++)ab(-+)c(++)d(++)e(--)bacf(+-) g(+-)h(+-)ei(-+)j(++)djfidhgihebgjfc	(29)0000
14	abcdaebdfghicjehjfgi	a(++)b(+-)bc(--)d(-+)e(++)bacdef(+-) g(--)h(+-)di(-+)aj(--)ijfehgcighjf	(29)0000

TABLE 2. The compact hyperbolic manifolds arising from a dodecahedron with dihedral angle $2\pi/5$ and an icosahedron with angle $2\pi/3$, [17].

$(\Gamma_f)\gamma$, where f is the common face of Σ and Δ, and $\gamma \in \Gamma_v$. Two faces $(\Gamma_f)\gamma_1$ and $(\Gamma_f)\gamma_2$ are identified by K exactly when $(\Gamma_f)\gamma_1 k = (\Gamma_f)\gamma_2$ for some $k \in K$. Similarly for the edge identifications–take cosets of Γ_e for e the common edge of Δ and Σ.

We will say that two X-manifolds M_1 and M_2, for $X = S^3, \mathbb{H}^3$ (respectively $X = \mathbb{E}^3$) are the same if and only if there is an X-isometry (resp. X-similarity) between them. Equivalently, if $M_i = X/K_i$, then the K_i are conjugate in the group of isometries (resp. similarities) of X. In the hyperbolic case, the following will help in distinguishing manifolds:

Theorem 3 (Margulis [3])**.** Let G be a connected semisimple Lie group with trivial centre and no compact factors, and Γ an irreducible lattice (discrete subgroup with finite Haar measure) in G. Then the commensurator $\text{comm}(\Gamma)$ is discrete if and only if Γ is non-arithmetic.

Arithmetic is meant here in the sense of [4], and the commensurator of Γ is the subgroup consisting of those $h \in G$ such that Γ and $h^{-1}\Gamma h$ are commensurable (have intersection of finite index in each). If we take $G = PO_{1,3}(\mathbb{R})$ to be the full isometry group of $\mathbb{H}^3$, then G has two connected components, one of which, $G^+ = PSO_{1,3}(\mathbb{R}) \cong PSL_2(\mathbb{C})$, consists of the orientation preserving isometries. Although G

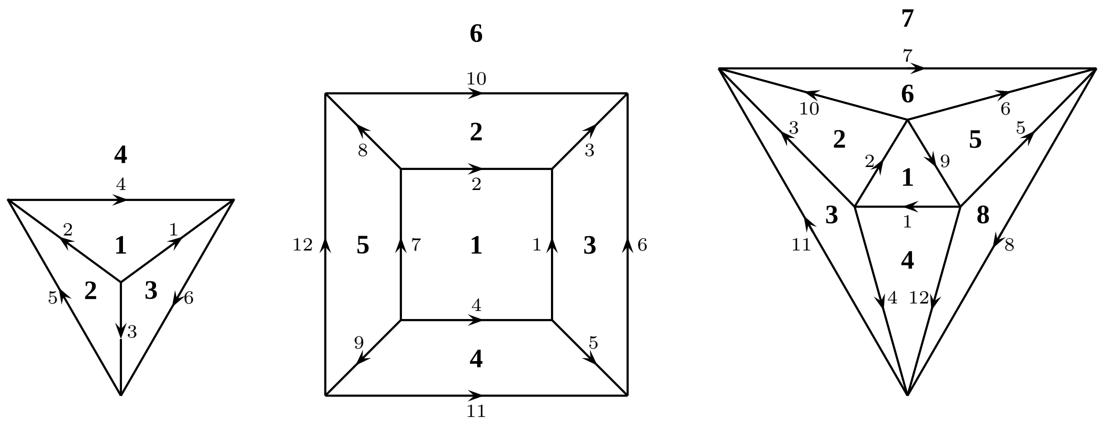


FIGURE 1.

is thus not connected, it is nevertheless easy to see that Theorem 3 holds for Γ in G. The arithmeticity of hyperbolic Coxeter groups is easily determined using the results of [19], from which we get in particular that the group with symbol

$$\circ\overset{5}{\text{———}}\circ\text{———}\circ\overset{6}{\text{———}}\circ, \tag{3}$$

is *non*-arithmetic. In [1] the six cofinite discrete subgroups of G having the smallest covolume are enumerated. In particular, they are all commensurable with the Bianchi groups $PGL_2\mathcal{O}_1$ or $PGL_2\mathcal{O}_3$, where $\mathcal{O}_d$ is the ring of integers in the number field $\mathbb{Q}(\sqrt{-d})$. By comparing volumes, one sees that if the group Γ with symbol (3) is not maximal, then it is contained in a $\overline{\Gamma}$ one of the six above, which cannot be, for the six above are arithmetic, while $\overline{\Gamma}$, being commensurable with Γ, is not.

Suppose then we have $K_i, i = 1, 2$, torsion free subgroups of the Γ with symbol (3), and a $g \in PO_{1,3}(\mathbb{R})$ such that $g^{-1}K_1g = K_2$. Then $g \in \text{comm}(\Gamma)$. By Theorem 3, $\text{comm}(\Gamma)$ is also discrete in G, and by the maximality of Γ, we have $\Gamma = \text{comm}(\Gamma)$. Thus $g \in \Gamma$. This reduces consideration of the conjugacy of the K_i in G (which is hard), to the much easier question of their conjugacy in Γ.

4. THE MANIFOLDS

Of the fifteen Platonic solids listed at the end of Section 1, four can be removed from consideration using Theorem 2, as the number of edges of Σ is not divisible by p. Of those that remain, the spherical dodecahedron with dihedral angle $2\pi/3$ was handled in [11] with results listed in Table 1 (the notation is described below). The first of the two manifolds is the Poincaré homology sphere. The compact hyperbolic dodecahedron and icosahedron with angles $2\pi/5$ and $2\pi/3$ were investigated in [17] with the results in Table 2–the first eight manifolds come from the dodecahedron, the remainder from the icosahedron[1]. The first is the Weber-Seifert space. This leaves the spherical $\{3,3,3\}, \{4,3,3\}$ and $\{3,4,3\}$; the Euclidean $\{4,3,4\}$ and hyperbolic $\{4,4,3\}, \{4,3,6\}, \{5,3,6\}$ and $\{3,3,6\}$.

As no doubt the reader has gathered by now, the only practical way the techniques of the previous section can be implemented is computationally. We use Sims's low index subgroups algorithm as implemented in Magma [6] to find the homomorphisms required by Theorem 1 when $X = \mathbb{E}^3$ and $\mathbb{H}^3$. For the spherical manifolds, we use Theorem 2. In any case, we obtain a complete list of the K, subgroups of

[1] It should be noted that while there are pairs in Table 2 with the same first homology, algebraic arguments are provided in [17] that show that the list is non-redundant (this is to be contrasted with the list in [2] which contains isometric pairs). Generally this involves consideration of quotients of terms in the derived series for $K = \pi_1(M)$, for instance, K'/K''.

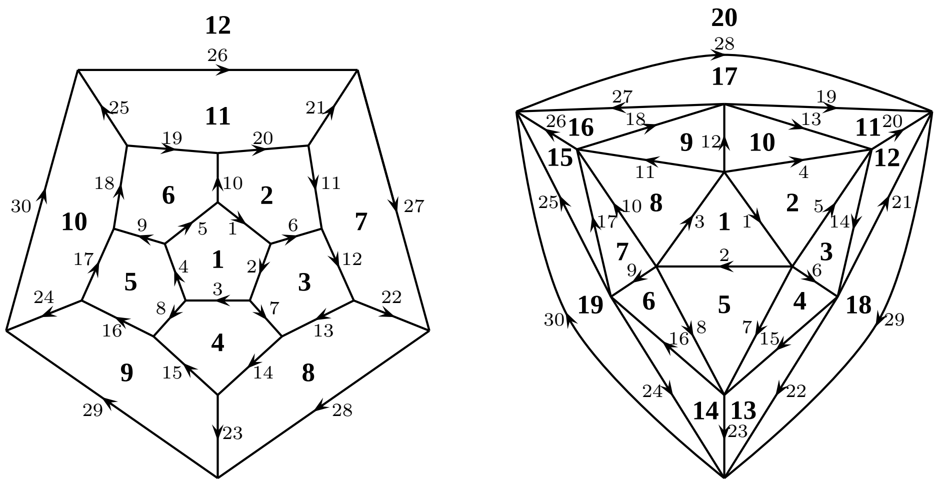


FIGURE 2.

the various Γ, satisfying the conditions of the two Theorems. As we want orientable manifolds, we also require that the generators of K are words of even length in the generators for Γ. The resulting K will be *non-conjugate in* Γ, although not necessarily so in G, the full isometry/similarity group of X.

The results are listed in Tables 3-5 which we will discuss in some detail presently. First we describe the notation. In each of the Tables, the column headed N indexes the manifolds M_i carrying the indicated geometric structure. The columns FI and EI give the face and edge identifications in the form of an encoded string of letters and $\pm$ signs to be read in conjunction with Figures 1-2. The i-th and j-th faces are paired when the i-th and j-th positions of the string in column FI are occupied by the same letter. Similarly for the edge identifications, where a string of $\pm$'s after a letter indicates whether the corresponding edge is identified with subsequent ones with the orientations matching or reversed. For example, the manifold M_{18} arising from the dodecahedron $\{5,3,6\}$ has edge identifications

```
a(+---+)b(+--++)bc(++--++)d(---+-)
bcae(+-+--)ceadddbeacedcaabecbed,
```

where `e` indicates that edges $9, 11, 17, 20, 26$ and 29 are identified, and `e(+-+--)` means edge 9 is identified with edge 11 so that the identifications match, with edge 17 so they are reversed, with edge 20 so they match, and so on. From the data in these two columns one may reconstruct the side pairing transformations $\{\gamma_S\}_{s\in\Sigma}$. In particular, the vertex identifications can be obtained in the spherical and Euclidean cases; in the hyperbolic there are no vertices! (they lie on the boundary of hyperbolic space in these non-compact examples). The next column in Table 5 gives the number of cusps. The final column gives the first homology $H_1(M_i,\mathbb{Z}) = \mathbb{Z}_a \oplus \mathbb{Z}_b \oplus \mathbb{Z}_c \oplus \mathbb{Z}_d \oplus \mathbb{Z}^e$ in the form of a sequence `abcde` (brackets are used in Tables 1-2 to distinguish double digits).

Table 1 gives the spherical results. Manifold M_1 comes from the tetrahedron in $\{3,3,3\}$, M_2 and M_3 from the cube in $\{4,3,3\}$ and M_4, M_5 and M_6 from the octahedron in $\{3,4,3\}$. Manifold M_3 is Montesinos's quaternoinic space [13, page 120] while M_6 is his octahedral space [13, page 117]. This leaves the issue of whether M_2 and M_5 are isometric, for they have the same homology. Now, M_2 arises from a subgroup K_2 of the group Γ with symbol,

$$\circ\overset{4}{\text{———}}\circ\text{———}\circ\text{———}\circ.$$

By Theorem 2(1), the order of K_2 is the index in Γ of Γ_v, and since $|\Gamma| = 2^4 4!$ (it is the Weyl group B_4) and $|\Gamma_v| = 48$, the number of symmetries of a cube, we have $|K| = 8$ (in fact it turns out that

N	FI	EI	H_1
1	abab	a(--)b(--)aabb	50000
2	ababcc	a(++)b(+-)aac(+-)bcd(+-)bcdd	80000
3	abcbca	a(++)b(--)c(+-)cd(--)bdabdac	22000
4	abcacbdd	a(++)b(+-)c(+-)ad(++)cbdacdb	26000
5	abcacdbd	a(++)b(-+)c(++)ad(-+)cbcaddb	80000
6	abcdcdab	a(++)b(++)c(++)d(++)bcdadabc	30000

TABLE 3. The spherical manifolds

N	FI	EI	H_1
7	abacbc	a(+++)b(+++)aac(+++)bccbcba	30001
8	abbcca	a(-+-)ab(--+)c(-+-)bacbbacc	22001
9	abccba	a(-+-)ab(--+)c(+--)bccbbcaa	44000
10	abcbca	a(+++)b(+++)c(+++)bcaaccbba	00003
11	abcbca	a(+++)b(+++)c(-+-)cbaacbbca	20001
12	abcbca	a(-+-)b(+--)c(+++)bcaaccbba	22001

TABLE 4. The Euclidean manifolds

N	FI	EI	C	H_1
13	ababcdcd	a(---)aaab(---)c(+-+)bccbcb	2	00002
14	abacbdcd	a(-+-)b(+--)babbaac(---)ccc	2	00002
15	ababcc	a(++---)b(-+-+-)aabbbbaaba	2	20002
16	ababcc	a(++--+)b(+++--)aabbbabbaa	1	24001
17	abcbca	a(+-+-+)b(----+)bbabaabaab	2	20002
18	abacbddceeff	a(+---+)b(+--++)bc(++--++)d(---+-) bcae(+-+--)ceadddbeacedcaabecbed	1	20001
19	abacdcdbefef	a(--+-+)b(+-+--)bc(+----)d(--+-+) bcacdcadddae(----+)badcaceeeebeb	2	20002
20	abacdbdcefef	a(-+-++)b(+--++)bc(+--+-)d(+---+) bcaadcadddce(----+)bcdacaeeeebeb	2	20002
21	abcacdedeffb	a(++---)b(--+++)abbc(+-++-)bbc d(--++-)dadbadde(---+-)ceecceeadedc	1	22001
22	abcacdedfebf	a(+--++)b(--+++)abbc(+-+++)bbc d(-+++-)e(+-+--)edbadedeadcceeaedac	1	22002
23	abcacdedfefb	a(+-++-)b(--+++)abbc(++++-)bbc d(-+-+-)e(+-+++)edbadecdacdceeaecad	1	26002
24	abcacdedeffb	a(+----)b(--+++)abbc(++-+-)bbc d(-++-+)ae(---+-)dbadecdecdceeeacad	2	22002
25	abcacbdeedff	a(+-+--)b(+-+++)ac(+--++)d(-++-+) e(--+-+)dbdedbccaebeadceecbabacd	2	60002
26	abcacdebdeff	a(+--++)b(+-+++)ac(-+-+-)d(-++--) e(+-+--)dbdecbccaebeacdeedbabadc	2	20002
27	abcbdefdcfae	a(-++--)b(--+++)ac(+++--)d(--+--) ccbdbdae(+++-+)daeeebaceccadebbd	1	22001

TABLE 5. The hyperbolic manifolds

$\pi_1(M_2) \cong \mathbb{Z}_8$ with generator $x_3x_2x_1x_4$, where x_i is the generator of Γ corresponding to the i-th node from the left in the symbol). On the other hand, by the same argument, the group K_5 yielding M_5 must have order 24 (Γ in this case is the Weyl group F_4 of order 1152). Thus, the two fundamental groups are not isomorphic, and so the manifolds are non-isometric.

Table 2 gives the Euclidean manifolds, with M_{10} the 3-torus. Unfortunately, we were not able to determine, by the techniques of this paper, whether M_8 and M_{12} were isometric or distinct.[2]

Table 3 gives the hyperbolic results. Manifolds M_{13} and M_{14} come from the octahedron in $\{4, 4, 3\}$, M_{15}, M_{16} and M_{17} from the cube in $\{4, 3, 6\}$ (see also [15]) and M_i, $i = 18$ to 27, from the dodecahedron in $\{5, 3, 6\}$. Manifold M_{14} is the Whitehead link complement [18, Section 3.3]. The tetrahedron in $\{3, 3, 6\}$ gave no orientable manifolds, although the non-orientable Gieseking manifold of 1911 is known to arise from it.

Manifolds M_{13} and M_{14} are non-isometric, despite having the same first homology, for, using low index subgroups in Magma again, K_{13} has five conjugacy classes of index 3 subgroups while K_{14} has six, so these two groups cannot be conjugate. For the same reason, M_{15} and M_{17} are distinct. Now the group $\Gamma = \{4, 3, 6\}$ is arithmetic by [19], and thus the subgroups K_{15} and K_{17} are too. On the otherhand, by the comments at the end of Section 3, K_{19}, K_{20} and K_{26} are non-arithmetic, so cannot be isomorphic to K_{15} and K_{17}. Hence M_{15} and M_{17} are not isometric to any of M_{19}, M_{20} or M_{26}.

Finally, there are a number of pairs with the same first homology among the M_i for $i = 18$ to 27. Clearly M_{22} and M_{24} must be distinct, for they have a different number of cusps. In fact, all ten are distinct: the corresponding K_i are non-conjugate in $\{5, 3, 6\}$ by construction, and then Theorem 3 and the comments at the end of Section 3 give that they are non-conjugate in $G = PO_{1,3}(\mathbb{R})$.

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF DURHAM, DURHAM DH1 3LE, ENGLAND
E-mail address: brent.everitt@durham.ac.uk
Current address: Department of Mathematics, University of York, York YO10 5DD, England
E-mail address: bje1@york.ac.uk

[2] I Prok [15] has shown that M_8 and M_{12} are related by a Euclidean similarity and so are indeed the same manifold.